Reduced- and mixed-precision finite element kernels

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1. Background

Floating point formats

Recent trend in chip manufacturing: Many new chips supporting low-precision computations. Double precision not prioritised due to AI focus.

Format	unit roundoff u		Range
fp64 (double)	2^{-53}	$\approx 1.11 \times 10^{-16}$	$10^{\pm 308}$
fp32 (single)	2^{-24}	$\approx 5.96 \times 10^{-8}$	$10^{\pm 38}$
fp16 (half)	2^{-11}	$\approx 4.88 \times 10^{-4}$	$10^{\pm 5}$
bfloat16 (half)	2^{-8}	$\approx 3.91 \times 10^{-3}$	$10^{\pm 38}$

Half vs double max speedups: $\times 4$ on CPUs, $\underline{\times 32+ \text{ on tensor-cores/AMX}}$.

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Today's focus

Designing efficient mixed-precision finite element cell kernels.

Motivation (see e.g., [Abdelfattah et al. 2021])

Many mixed-precision algorithms exploit reduced-precision operators for speedup: Preconditioning, linear/nonlinear solvers, and timestepping methods.

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Warning: Work in progress!

AMX accelerators on Intel Xeon CPUs

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AMX implement pairwise matrix-matrix multiply and accumulate as follows:

$$S \mathrel{+}= AB + CD, \quad \{A, C\} \subset \mathbb{R}^{m \times k}, \ \{B, D\} \subset \mathbb{R}^{k \times n},$$

where A, B, C, and D are stored in bf16 and S in single precision. Operations carried out in single. Max sizes: $m, k, n = 16$.

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Of course, AMX cannot be used for everything.

2. Mixed-precision finite element kernels

Challenge: Exploiting MP accelerators in FE cell kernels

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The Masterplan

A) **Make mat-mat products the bottleneck.**
B) **Rounding error analysis.**
C) **Implementation.**

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Example: Poisson form over a single cell $K \subset \mathbb{R}^d$,

$$a_K(u_h, v_h) = \int_K \nabla u_h \cdot \nabla v_h \,\mathrm{d}x, \quad u_h, v_h \in V_h|_K = \mathrm{span}(\{\phi_i\}_{i=1}^m).$$

Cell kernels: Compute local matrix resulting from the above form (can also do actions):

$$A_{ij} = a_K(\phi_j, \phi_i), \quad A \in \mathbb{R}^{m\times m},$$

A) Cell kernels as sum of mat-mat products

It can be shown that A can be expressed as a sum of triple matrix products:

$$A = \sum_s \sum_t B_s D_{st} B_t^T, \quad B_s \in \mathbb{R}^{m \times n_q}, \quad D_{st} \in \mathbb{R}^{n_q \times n_q},$$

$B =$ Derivatives of basis functions at quadrature points (3D tensor).
$D =$ Quadrature weights and geometry at quadrature points (4D sparse tensor).

For actions: Use cell batching $\implies$ Also obtain sum of mat-mat prods.

B) Rounding error analysis

Theorem (proof in progress)

Computing A using the following precisions:

- u_{store} for storing B and computing the action of D,
- u_q for performing and accumulating mat-mat products,
- u_g for computing the geometry tensor,
- u_p for evaluating basis functions on the reference cell,

yields instead $\hat{A}$ satisfying

$$\|A - \hat{A}\|_\infty \lesssim \left(u_{\text{store}} + (d^2 + n_q)u_q + \kappa_\infty(J)u_g + p^d u_p \right) \|A\|_\infty.$$

B) Rounding error analysis

Cost-accuracy trade off. Want to use low precision for speed, but stay accurate.

Objective: Obtain $O(u)$ accuracy where u is the half-precision unit roundoff.

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AMX/tensor core strategy:

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AMX/tensor core strategy:

1. High-precision for geometry and basis evaluation.

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AMX/tensor core strategy:

1. High-precision for geometry and basis evaluation.
2. Cast to half and use AMX for the rest.

C) Implementation (C++)

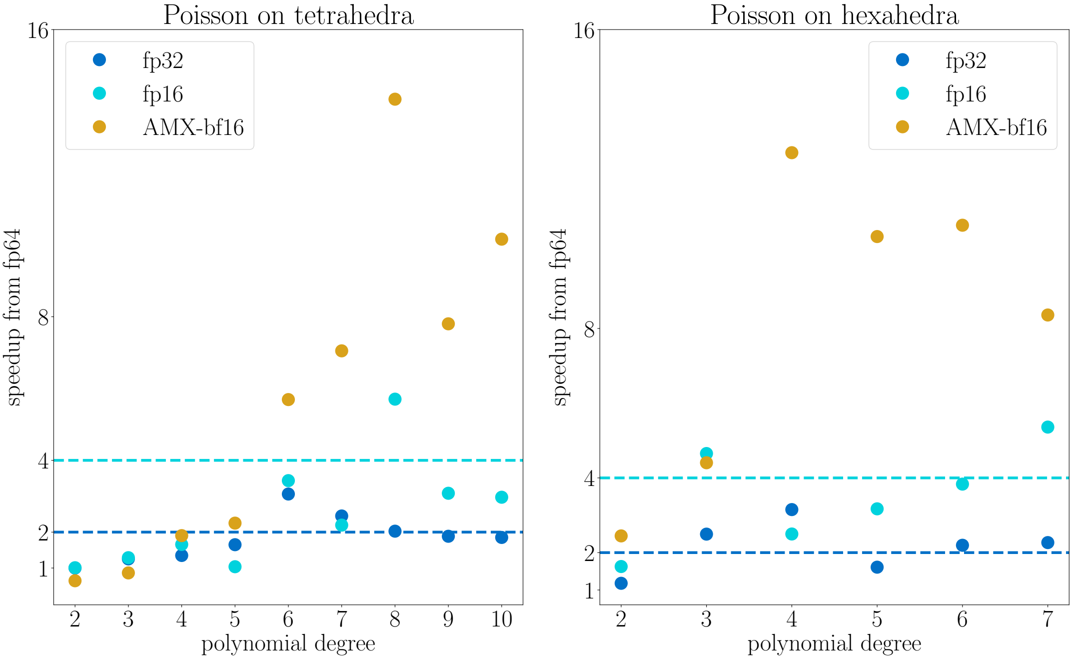
Implementation challenges

1. Brand new chip functionalities mean lack of compiler support and bugs.
2. Compilers won't use AMX for you. Exotic system calls and Intel intrinsics needed.
3. AMX library support limited/bugged, yet growing.

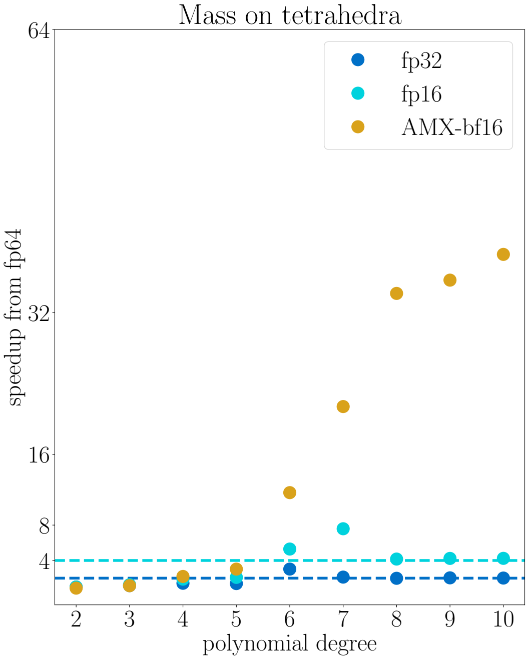
Testing and debugging: Use FFCX-generated kernels.

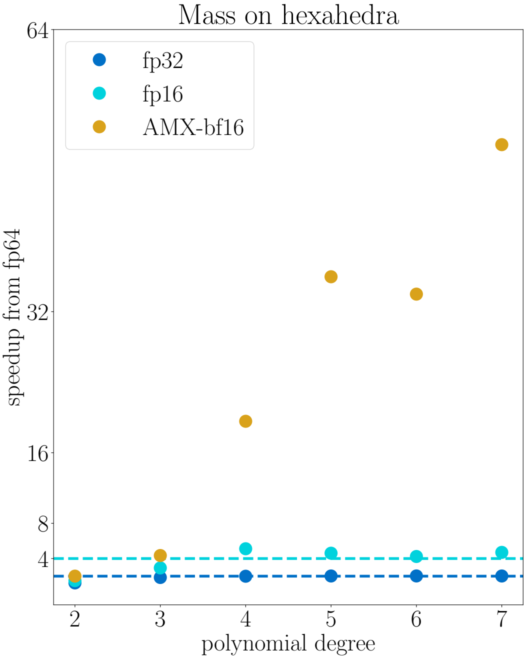
3. Numerical results

Numerical results - Poisson form in 3D



Numerical results - Mass form in 3D





4. Conclusions

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To sum up

- We can compute FE kernels up to $10 - 50\times$ faster using AMX. Implementation was admittedly challenging. Luckily, AMX software support is growing.
- Multiple applications can benefit from faster FE kernels: Preconditioning, iterative refinement, inexact Krylov and Newton solvers, timestepping methods, etc.
- GPU kernels and algorithmic applications will be addressed in future work.

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Thank you for listening!

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More info about me and my work at: `croci.github.io`

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